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What is MHD?

short for MagnetoHydroDynamics

The study in which the behavior of a lot of charged particles (plasmas) is treated as motion (dynamics) of a group (fluid) taking interaction between the group and magnetic/electric field into account.

Hydrodynamics;
The fluid consists of a lot of particles. Each particle is in the different position in the real space and the velocity space. When we study the properties, we do not care each particle's property and we characterize it by using "density", "temperature", "pressure", "velocity", "electric charge density" and "current" as the averaged value with some kind of "weigh". And we analyze the averaged values when we study the fluid properties.

"MHD equilibrium and stability" means the force balance and the stability from view point of "MagnetoHydroDynamics"
What is the study of MHD equilibrium and stability?

**MHD equilibrium study;**
Does plasma move or not when plasma is softly put in the bottle made of the magnetic field? Is the bottle crushed?
What is the condition for plasma not to move?

**MHD stability study;**
Does plasma in the bottle made of the magnetic field move or not when plasma is slightly pushed? Does the whole of it or the part of it moves? What is the condition for plasma to stay?

Moving plasma leads to the source of the magnetic field
=> Situation is very complicated.
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MHD equil. study = Study of Bottle of mag. field (mag. configuration)

Trace of mag. field line =>
Closed line in 2 dimension

Coil of LHD (purple)
& plasma (yellow)

10 times poloidally rotating during 1
toroidal rotating

1 field line makes
a "basket"
Field lines make nested
"baskets"
=> magnetic surfaces

Nested baskets made of mag. field
lines keep away plasma from wall
=>
Good magnetic surfaces improve
insulation between plasmas and wall
Density and temperature grad in mag. bottle induce current

\[ j_i = q_i e v_i \times 2 \delta n \]  
\[ \delta n = (dn/dr) \delta r \]
\[ M_i v_i^2 / \delta r = q_i e v_i B \] (centrifugal vs electromag. force)

\[ j_i = (1/B)(2M_i v_i^2)(dn/dr) \]

In increases of temp., \( \delta v_i \) exists instead of \( \delta n_i \).

\[ \Rightarrow \] Current flows (cf; \( T_i = M_i v_i^2 / 2 \)).

For electron, similar current flows (cf; \( p = (n_e T_e + n_i T_i) \)).

\[ \Rightarrow \] Total current;

\[ j \propto \frac{1}{B} \frac{\partial p}{\partial r} \]

Current direction; reduces the original mag. field

\[ \Rightarrow \] Diamagnetic current

Diamag. current; orthogonal to both B and grad p \( \Rightarrow j \times B = \text{grad } P \)

\[ \Rightarrow \] Diamag. current; source of changing mag. field from vacuum
cf. Relationship between density, temp., press. and velocity of plasma (I)

In thermal equilibrium state (collisional and steady state);
Distribution func. is isotropic, and gauss func. in the velocity (Maxwell distribution func.)
The dispersion is defined by $T/m$, the average velocity is by $u$,

$$f \propto \exp\left(\frac{m(v-u)^2/2}{T}\right); \text{ Boltzmann constant Omitted}$$

$$N \equiv \int f dv; \text{ Total number of particles is defined by } N$$

$$\Rightarrow f = N\left(\frac{m}{2\pi T}\right)^{1.5} \exp\left(\frac{m(v-u)^2/2}{T}\right)$$

$$\int m(v-u)^2/2 f dv \Rightarrow \frac{3}{2} NT$$

Energy of particles in the moving coordinate system with $u$ (thermal energy)
For simplicity, we consider the pressure under the assumption $u=0$

The momentum which 1 particle gives a wall

$$2mv_x$$

The number of particles per unit of time which reach the wall with area size $S_x$,

$$v_x S_x n \quad (n \text{ denote density})$$

Then, the momentum per unit of time (force) by which the wall with area size $S_x$ is pushed,

$$2mv_x^2 S_x n$$

The pressure, $p$, corresponds to the above force divided by area size, (here isotropic pressure is assumed)

$$p=2mn<v^2>/3=nT$$

=> Pressure is density times temp.
Diamag. current in the presence of non-uniform mag. field

\[
\dot{j}_R = \frac{\nabla p}{B}
\]

Here non-uniform mag. field is assumed as \( B = R_0 B_0 / R \).

\[
j_R = R \frac{\nabla p}{R_0 B_0}
\]

Current density increases as more torus outwardly

Divergence of \( j \) is as the below

\[
div \ j = \frac{1}{R} \frac{\partial}{\partial R} \left( R \times R \frac{\nabla p}{R_0 B_0} \right) \approx \frac{2 \nabla p}{RB} \neq 0
\]

\[
\frac{d\sigma}{dt} + div \ j = 0 \quad \Rightarrow \quad \frac{d\sigma}{dt} \neq 0
\]

The current along mag. field line should be induced

\( \Rightarrow \)

Pfirsh-Schluter (PS) current

The charge appears at torus top and bottom (\( d\sigma/dt = 0 \))

Time evolution of charge (due to the connection of top and bottom) is necessary to satisfy the charge conservation
Evaluation of PS current (From $\nabla \cdot j = 0$ )

\[
\begin{align*}
\mathbf{j}_\parallel & \equiv \left( \frac{j \cdot \mathbf{B}}{B^2} \right) \mathbf{B}, \quad \nabla \cdot \mathbf{j}_\parallel = \nabla \cdot \left( \frac{j \cdot \mathbf{B}}{B^2} \right) \mathbf{B} = \mathbf{B} \cdot \nabla \left( \frac{j \cdot \mathbf{B}}{B^2} \right) = B \frac{\partial}{\partial s} \left( \frac{j \cdot \mathbf{B}}{B^2} \right) \sim \frac{\partial j \cdot \mathbf{B}}{\partial s}, \\
\mathbf{j}_\perp & \equiv \frac{B^2}{r} \mathbf{j} - \left( \frac{j \cdot \mathbf{B}}{B^2} \right) \mathbf{B} = \frac{B \times \nabla p}{B^2}, \\
\nabla \cdot \mathbf{j}_\perp & = \nabla \cdot \left( \frac{B \times \nabla p}{B^2} \right) = \nabla p \cdot \nabla \times \left( \frac{1}{B^2} \mathbf{B} \right) \\
& = \nabla p \cdot \left( \frac{2}{B^3} \nabla B \times \mathbf{B} \right) = -2 \nabla p \cdot \left( \frac{\nabla B \times \mathbf{b}}{B^2} \right) \\
\nabla p & \sim \hat{r} \frac{\partial p}{\partial r}, \quad \hat{\theta} \cdot \nabla B \sim B_0 \frac{\partial}{r \partial \theta} (1 - \varepsilon \cos \theta), \quad \frac{\partial}{\partial s} = \frac{\partial \theta}{\partial s} \frac{\partial}{\partial \theta} = \frac{t}{2 \pi R_0} \frac{\partial}{\partial \theta} \\
\nabla \cdot \mathbf{j}_\parallel & = -\nabla \cdot \mathbf{j}_\perp \implies \frac{\partial j \cdot \mathbf{B}}{\partial s} = 2 \nabla p \cdot \left( \frac{\nabla B \times \mathbf{b}}{B^2} \right) \\
\implies \quad \frac{t}{2 \pi R_0} \frac{\partial j \cdot \mathbf{B}}{\partial \theta} & \sim \frac{\partial p}{\partial r} \frac{1}{R_0 B_0} \sin \theta \implies j \cdot \mathbf{B} \sim -\frac{2 \pi}{t B_0} \frac{\partial p}{\partial r} \cos \theta
\end{align*}
\]

Current along mag. field line (PS current) increases with press. grad. and decreases with rotational transform, $t$, and magnetic field strength, $B_0$. 

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Plasma consists of a lot of charged particles (MHD equil. picture based on each particle's motion)

**Basis of behavior of charged particles; Drift**

Charged particle follows gyro-orbit along the magnetic field line in uniform mag. field w/o elec. field. In non-uniform mag. field and/or with elec. field, it makes the additional motion in perpendicular to mag. field (drift). This is the basis to understand the charged particle behavior.

### Bx∇B drift

Ion moves in the direction to $\textbf{B} \times \nabla \textbf{B}$ due to change of gyro-radius during gyro-motion.

Direction of elec. drift is opposite.

### ExB drift

Ion moves in the direction to $\textbf{E} \times \textbf{B}$ due to change of velocity during gyro-motion.

Direction of elec. drift is same.
Change of mag. bottle due to plasma I  -- based on particle motion --

Plasma cannot be confined by only toroidal mag. field (it moves even when plasma is softly put).

Reason
(1) Charge separation occurs due to $B \times \nabla B$ drift.
(2) The charges induce Elec. field $\Rightarrow$ Both ion and elec. move to torus outwardly due to $E \times B$ drift.

A countermeasure
Add the poloidal mag. field ($B_p \neq 0$) to connect the separated charges in torus-top and bottom due to the mag. filed line.

Elec. field is reduced, which suppresses $E \times B$ drift.

How to produce $B_p$
Tokamaks; toroidal current is induced.
Heliotron/Helical; external coils are helically wound.

Elec. easily moves because it is light to cancel separated charges.
Change of mag. bottle due to plasma II -- mag. axis moves --

Suppression of ExB drift due to production of $B_p$ (Suppression of charge separation)

$\Rightarrow$ Pfirsh-Schuter current (Equilibrium current) is induced
$\Rightarrow$ Mag. axis torus outwardly shifts due to vertical field by PS current $\Rightarrow$ Change of mag. bottle

Large $B_p$ $\Rightarrow$ Large pitch of mag. field line (rotational transform $\iota$) $\Rightarrow$ Small toroidal component of PS current $\Rightarrow$ Small Shafranov shift

Large plasma pressure (Large $\frac{dp}{dp}$) $\Rightarrow$ Large PS current $\Rightarrow$ Large Shafranov shift
Summary of PS current (from viewpoints of different aspects)

From the viewpoint of fluid and particle, driving mechanism of PS current is reviewed. From both viewpoints, it is indispensable to confine finite pressure plasma.

1. In toroidal mag. configuration (grad B /=0), only diamag. current does not satisfy div \( j = 0 \) condition and charges increase both in the torus top and the bottom. The finite poloidal mag. field in addition to toroidal field is necessary to satisfy div \( j = 0 \) and PS current appears.

2. In toroidal mag. configuration (grad B /=0), charge separation occurs due to Bx \( \nabla B \) drift. In order to suppress ExB drift due to the charge separation, the poloidal mag. field (\( B_p \neq 0 \)) is necessary to connect the torus-top and bottom of the mag. filed line. When the charges are cancelled, a current flows in the opposite direction inside and outside of torus, which is PS current's another aspect.
How is poloidal field produced — Helical and Tokamak —

External coils (helical coils) produces poloidal mag. field.
More suitable for steady state operation than tokamak
Construction is difficult because of complicated structure and needs of high accurate alignment.
Japanese scientist poroposed this concept.

Toroidal current produces poloidal mag. field.
For steady state operation, innovative concept on stationary toroidal current drive is necessary.
Construction is rather easy because of simpler structure than helical.
Why can helical coils produce $B_p$ (finite poloidal field)?

$B_\theta \sim \cos(L \theta - M \phi)$, $B_\phi \sim B_{\phi 0} [1 - \delta \cos(L \theta - M \phi)]$

$L$: poloidal pole number, $M$: toroidal period

In the above figure, $L=2$.

During $B_\theta > 0$, $B_\phi < B_{\phi 0}$
During $B_\theta < 0$, $B_\phi > B_{\phi 0}$

$\Rightarrow$ During $B_\theta > 0$, mag. field line proceeds slowly in toroidal direction. During $B_\theta < 0$, mag. field line proceeds fast.

This phase is shorter than the previous phase.

$\Rightarrow$
Mag. field line does not reach the starting point of the previous phase.

$\Rightarrow$
Mag. field line proceeds in the poloidal direction.
MHD equilibrium equation I

Starting from MHD equations

\[
\begin{align*}
\rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} &= \mathbf{j} \times \mathbf{B} - \nabla p, \\
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0, \\
\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \left( \frac{p}{\rho^\gamma} \right) &= 0, \\
\mathbf{E} + \mathbf{v} \times \mathbf{B} &= \eta \mathbf{j}, \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{j}, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0.
\end{align*}
\]

Motion eq.  
Continuity eq.  
State eq.  
Ohms low  
Maxwell eq.

Here we consider the MHD equilibrium (\( \partial / \partial t = 0 \)) under the assumption \( \mathbf{v} \ll \mathbf{v}_{\text{th}} \) (\( \mathbf{v}=0 \)). (This assumption is valid in typical fusion plasma)

\[
\begin{align*}
0 &= \mathbf{j} \times \mathbf{B} - \nabla p, \\
0 + 0 &= 0, \\
0 + 0 &= 0, \\
\mathbf{E} + 0 &= \eta \mathbf{j}, \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{j}, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0.
\end{align*}
\]

MHD equilibrium equation

\[
\begin{align*}
\mathbf{j} \times \mathbf{B} &= \nabla p, \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{j}, \\
\nabla \cdot \mathbf{B} &= 0.
\end{align*}
\]
MHD equilibrium equation II

Starting from MHD equil. eq.

\[ \mathbf{j} \times \mathbf{B} = \nabla p, \]
\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \]
\[ \nabla \cdot \mathbf{B} = 0. \]

\[ \mathbf{j} - \frac{\mathbf{B}}{B} \frac{\mathbf{B}}{B} = \mathbf{j}_\perp = \frac{\mathbf{B} \times \nabla p}{B^2}, \]
\[ \nabla \cdot \mathbf{j} = \nabla \cdot (\nabla \times \mathbf{B}) = 0. \]

1. Mag. field line and current lie on contour of pressure.
2. Contours of pressure coincide with mag. surfaces.

\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \Rightarrow \mathbf{B} = \nabla \psi \times \nabla \alpha \]
(Clebsch expression)

1. Both \( \nabla \psi \) and \( \nabla \alpha \) are orthogonal to mag. field line.
2. When contour of \( \psi \) is defined by mag. surface, \( \alpha = \text{const} \) on mag. surf. denotes magnetic field line.
Ex. of change of mag. bottle (mag. structure) due to plasma press.

Plasma press. induces current => changes confinement mag. field => changes shape of mag. bottle.

In LHD discharges

Low plasma press.

High plasma press.

Theoretical prediction

Vacuum mag. field

Mag. surf. in Vac.

OFMS in Vac.

$T_e (keV)$

$n_e (10^{19} m^{-3})$

$B_0 = 1.5 T$, $\langle \beta_{dia} \rangle \approx 0.2\%$

$B_0 = 0.5 T$, $\langle \beta_{dia} \rangle \approx 2.9\%$
プラズマ中にはいろんな電流が流れる

1. 反磁性電流
2. Pfirsh-Schluter(フィルシュ.シュルター)電流
   (MHD平衡が成り立つために必要な電流として説明済み)
3. オーミック電流
4. ビーム駆動(大河)電流
5. ブートストラップ電流
6. その他(電磁波駆動電流など)

これらの電流が、プラズマ閉込め容器(磁場配位)の形状を変化させる
=> その影響を定量的に調べることがMHD平衡研究
閉込め容器の形状が変わればプラズマの閉込め特性も影響を受けるので、MHD平衡研究は基盤的な研究。
3. オーミック電流

$$d\Phi/dt = R I_p$$

4. ビーム駆動(大河)電流

米国、GA社の大河博士が予言

水素イオンと電子の衝突の差により電流が流れる
5.ブートストラップ電流の駆動機構 (I)

磁場強度に強弱がある時の荷電粒子の運動

電場がゼロで磁場がゆっくり変化している場の荷電粒子は、運動エネルギーの他に磁気モーメントが保存する。

\[
\frac{m v_{\|}^2}{2} = E - \mu_m B \quad \text{より}
\]

\[
E < \mu_m B_{\text{max}} \quad \text{の粒子は} B = B_{\text{max}} \quad \text{の領域に到達できない。}
\]

\[
\Rightarrow \frac{m v_{\|}^2}{2} + \frac{m v_{\perp}^2}{2} < \frac{m v_{\perp}^2}{2} B_{\text{max}} B_{\text{min}} \]

\[
\Rightarrow z = 0 \quad \frac{v_{\perp}}{v_{\perp 0}} < \sqrt{\frac{B_{\text{max}}}{B_{\text{min}}} - 1} \quad \text{の荷電粒子は}
\]

\[
B = B_{\text{max}} \quad \text{の領域に到達する前に磁場の弱い方へ反射される。}
\]

\[
\Rightarrow \text{捕捉粒子と呼ばれる粒子が存在する。}
\]
5.ブートストラップ電流の駆動機構 (II)

磁場の弱いところに捕捉される捕捉粒子の存在と密度、温度勾配によりトロイダル方向に余剰のモーメンタム源が現れる。

圧力勾配が駆動する電流 "ブートストラップ電流"が流れる。
ヘリカルコイルで回転変換（磁場の捩じれ）が生じる理由

\[ B_{\theta} \sim \cos(L\theta-M\phi), \quad B_{\phi} \sim [1-\delta \cos(L\theta-M\phi)] \]

\( L, M \) はそれぞれポロイダル局数, トロイダル周期数

図の例では, \( L=2 \).

\( B_{\theta} \) が正の間は \( B_{\phi} \) が1より小さく, \( B_{\theta} \) が負の間は \( B_{\phi} \) が1より大きい. つまり, \( B_{\theta} \) が正の間は磁力線は \( \phi \) 方向にあまり進まず, \( B_{\theta} \) が負の間に磁力線は早く前に進む.

こちらのほうが短い, その前の半周期で \( \theta \) 方向に進んだ分, 戻ってこない

\[ \Rightarrow \]

磁力線は \( \theta \) 方向に進む
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What drives MHD instabilities in magnetized plasmas?

Two driving mechanism are considered.
(1) Pressure gradient (pressure driven mode)
(2) Plasma current (current driven mode)

(1) =>
appears in both helical and
tokamak plasmas.
# Interchange/Ballooning mode
(2) =>
appears in only tokamak plasmas.
# Kink/Tearing mode

MHD Stable on not?

Does plasma in mag. bottle move or not when plasma is slightly pushed?
Does the whole of it or the part of it moves?
Physical picture of pressure driven instabilities

Unstable conditions;
Pressure increases as magnetic field strength increases.

Physical picture of instability
(1) Density perturbation appears
(2) Direct. of density gradients coincide with mag. field strength gradient (Bad curvature).
=> Charges separates due to $\mathbf{B} \times \nabla \mathbf{B}$ drift.
(3) Charge separation due to $\mathbf{E} \times \mathbf{B}$ drift enhances density perturbation.

Unstable conditions; Pressure increases as magnetic field strength increases.

Physical picture of instability
- Density perturbation appears
- Direct. of density gradients coincide with mag. field strength gradient (Bad curvature).
- Charges separate due to $\mathbf{B} \times \nabla \mathbf{B}$ drift.
- Charge separation due to $\mathbf{E} \times \mathbf{B}$ drift enhances density perturbation.
Plasma consists of a lot of charged particles (MHD equil. picture based on each particle's motion)

**Basis of behavior of charged particles; Drift**

Charged particle follows gyro-orbit along the magnetic field line in uniform mag. field w/o elec. field. In non-uniform mag. field and/or with elec. field, it makes the additional motion in perpendicular to mag. field (drift). This is the basis to understand the charged particle behavior.

\[
\frac{mv^2}{r} = qvB \Rightarrow r = \frac{mv}{qB}
\]

**Bx∇B drift**

Ion moves in the direction to \(B \times \nabla B\) due to change of gyro-radius during gyro-motion.

Direction of elec. drift is opposite.

**ExB drift**

Ion moves in the direction to \(E \times B\) due to change of velocity during gyro-motion.

Direction of elec. drift is same.
Here we consider a unstable system as analogs of the pressure driven instability.

**gravitational instability**

**destabilizing term;**
gravitation and deference of weight between 2 fluids

**stabilizing term;**
Nothing

**pressure driven instabilities in plasma**

grad B drift force (magnetic well/hill)
deference of pressure (pressure gradient)

field line bending (magnetic shear)
Charge separation due to $\mathbf{E} \times \mathbf{B}$ drift enhances density perturbation

# In the rational surface resonated with the wave number of dens. fluc., separated charges cannot be canceled.

$\Rightarrow$ Den. fluc. with resonated wave numbers grows.

(Unstable)

Rational surface resonated with dens. fluc.

Rotational transform (pitch of mag. field line winding) = $m/n$

$\Rightarrow$ When mag. field line turns toroidally $n$ times, it turns poloidally $m$ times.

Wave number of $m$ and $n$ (toroidal and poloidal mode num.)

$\Rightarrow$ When fluc. goes around toroidally $n$ times, it turns poloidally $m$ times, min. and max. of the amp. of fluc. coincides each other before and after.
Physical picture of pressure driven instabilities IV

The effect of the magnetic field shear in the radial direction on the ideal interchange mode

In order that the perturbation grows over the resonant surface, it is necessary to bend the magnetic field line so that it makes the direction of the field line same with that of the resonant surface.

Unless it bends the field line, the electron motion on the next surfaces cancels the separated charges for the ExB drift.

The electron moving on a rational surface returns to the exact same position after $n$ toroidal turns. There the charge cancellation due to the electron does not occurs for the resonant modes

$rational\ surface; \ \imath=n/m.$

$rational\ surface; \ \imath=n/m.$

$\imath; \ rotational\ transform =1/q.$

$n, m; integer.$

$⇒$

$Mag.\ shear\ has\ the\ stabilizing\ effet.$

Minor radial outward direction
Physical picture of pressure driven instabilities V

What is the effect of resistive in pressure driven instability?

*Hint!*

Charge separation due to $\mathbf{E} \times \mathbf{B}$ drift enhances density perturbation

# In the rational surface resonated with the wave number of dens. fluc., separated charges cannot be canceled.

$\Rightarrow$ Den. fluc. with resonated wave numbers grows. (Unstable)

Expands to slab.
Here we consider several possible mechanical system as analogs of the MHD stability.

In Fig.(a), if the ball is moved a small distance from equilibrium position, it simply oscillates around this point. Even though the ball never returns to rest at its equilibrium position, this status is “stable”. In Fig.(b), a small perturbation off the top of the hill sets the ball rolling further away from its equilibrium position, this status is “unstable”.

There are two methods to analyze the MHD stability.
(1) Analyzing the time evolution of the displacement, $\xi$, in MHD equations, especially momentum equation.
(2) Analyzing the change of the potential energy when a displacement occurs. Since the total energy is conserved in the frictionless system, when the kinetic energy increases, the potential energy decreases. When the potential energy decreases, the system is unstable.

Here only linear stabilities are considered.
Linearized MHD Eq. I

Starting from ideal MHD equations

\[ \rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = \mathbf{j} \times \mathbf{B} - \nabla p, \]
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \]
\[ \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \left( \frac{p}{\rho'} \right) = 0, \]
\[ \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0, \]
\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0. \]
Here some quantities are linearized (separate the equilibrium part (suffix 0) and the perturbed part (suffix 1).).

\[ \mathbf{v}(\mathbf{r}, t) = \mathbf{v}_0(\mathbf{r}) + \mathbf{v}_1(\mathbf{r}, t), \mathbf{v}_0(\mathbf{r}) = 0, \quad \mathbf{j}(\mathbf{r}, t) = \mathbf{j}_0(\mathbf{r}) + \mathbf{j}_1(\mathbf{r}, t), |\mathbf{j}_1| << |\mathbf{j}_0|, \]
\[ \mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r}, t), |\mathbf{B}_1| << |\mathbf{B}_0|, \quad \mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r}) + \mathbf{E}_1(\mathbf{r}, t), \mathbf{E}_0(\mathbf{r}) = 0, \]
\[ \rho(\mathbf{r}, t) = \rho_0(\mathbf{r}) + \rho_1(\mathbf{r}, t), \rho_1 << \rho_0, \quad p(\mathbf{r}, t) = p_0(\mathbf{r}) + p_1(\mathbf{r}, t), p_1 << p_0. \]

The 1st order momentum equations are as follows:

\[ \rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \mathbf{j}_1 \times \mathbf{B}_0 + \mathbf{j}_0 \times \mathbf{B}_1, \quad \frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_1) = 0, \]
\[ \frac{\partial p_1}{\partial t} = -\mathbf{v}_1 \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \mathbf{v}_1, \quad \mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_0 = 0, \]
\[ \nabla \times \mathbf{B}_1 = \mu_0 \mathbf{j}_1, \quad \nabla \times \mathbf{E}_1 = -\frac{\partial \mathbf{B}_1}{\partial t}, \quad \nabla \cdot \mathbf{B}_1 = 0. \]
Linearized MHD Eq. II

Summarizing the above equations,

\[ \rho_0 \frac{\partial^2 v_1}{\partial t^2} = -\nabla \{ v_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot v_1 \} + j_0 \times \{ \nabla \times (v_1 \times B_0) \} + \frac{1}{\mu_0} [\nabla \times \{ \nabla \times (v_1 \times B_0) \}] \times B_0. \]

When \( v_1 \) replaces a Lagrangian variable, \( \partial \xi / \partial t \),

\[ \rho_0 \frac{\partial^2 \xi}{\partial t^2} = F(\xi), \]

\[ F(\xi) = -\nabla \{ \xi \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \xi \} + \frac{1}{\mu_0} (\nabla \times B_0) \times Q + \frac{1}{\mu_0} (\nabla \times Q) \times B_0. \]

where \( Q \equiv \nabla \times (\xi \times B_0) \).

In the linear stability analysis, the following expression of the time evolution of the perturbation is useful,

\[ \xi(r, t) = \xi(0) \exp(-i \omega t). \] If \( \omega \) is imaginary, the mode grows (unstable).

Then

\[ -\rho_0 \omega^2 \xi = F(\xi). \]

Here it should be noticed that \( F \) is a self-adjoint operator,

\[ \int dV x \cdot F(y) = \int dV y \cdot F(x). \]

\[ \frac{1}{2} \rho_0 \omega^2 \int dV \xi^* \xi = -\frac{1}{2} \int dV \xi^* F(\xi), \]

\[ \frac{1}{2} \rho_0 \omega^2 \int dV \xi^* \xi = \frac{1}{2} \rho_0 \omega^2 \int dV \xi^* \xi^* = -\frac{1}{2} \int dV \xi F(\xi^*). \]

\[ \Rightarrow \frac{1}{2} \rho_0 \omega^2 \int dV \xi^* \xi = \frac{1}{2} \rho_0 \omega^2 \int dV \xi^* \xi^* \Rightarrow \omega^2 = \omega_*^2. \]

\[ \Rightarrow \omega^2 \text{ should be real.} \]

Here \( * \) denotes a complex conjugate.
Because $K$ is positive, the sign of $\delta W$ determines the stability of the system. $\delta W > 0$ => stable. $\delta W < 0$ => unstable.

Here $K$ and $\delta W$ correspond to the kinetic energy and the potential energy.

After some calculations, $\delta W$ is rewritten as

$$\delta W = \frac{1}{2} \int dV \left[ \rho_0 (\nabla \cdot \xi)^2 + (\xi \cdot \nabla \rho_0)(\nabla \cdot \xi) + \frac{|Q|^2}{\mu_0} \right] j_0(Q \times \xi)$$

Change of the internal energy of plasma without magnetic energy

Change of the magnetic energy

Work against the unbalanced magnetic force
Linearized MHD Eq. IV

\[ \delta W = \frac{1}{2} \int_{\text{plasma}} \left[ \frac{Q_\perp^2}{\mu_0} + \frac{B_0^2}{\mu_0} \left( \nabla \cdot \xi_\perp + 2\xi_\perp \cdot \kappa \right)^2 + \gamma p_0 (\nabla \cdot \xi)^2 - 2(\xi_\perp \cdot \nabla p_0)(\xi_\perp \cdot \kappa) - \mathbf{j}_\parallel \cdot (Q_\perp \times \xi_\perp) \right] \]

- shear alfven wave (line bending term)
- sound wave of plasma
- compressional alfven wave
- stabilizing term
- current driven destabilizing term
- pressure driven destabilizing term

When the mode is localized \( k_\perp a >> 1 \), pressure driven mode is dominant.

Current driven mode;
The global mode is more easily unstable than the localized mode.
\( k_\perp a \sim 1 \)
Ideal Interchange mode I ---Reduced MHD equation---

When the high aspect approximation ($\varepsilon = a/R_0 << 1$) and the high beta ordering ($\beta \sim \varepsilon$) are applied to the full MHD equations, in a quasi toroidal coordinates, $(r, \theta, \phi)$, the following reduced MHD equation is obtained. [$\xi = \text{grad } U \times z \ (\xi = mU/r \omega)$]

$$\frac{\partial \psi}{\partial t} = \mathbf{B} \cdot \nabla U, \left( \frac{\partial}{\partial t} + \mathbf{B} \cdot \nabla \right)p = 0, \quad \rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right)\nabla^2 U = -\mathbf{B} \cdot \nabla j_\phi - \hat{\phi} \times \kappa_r \cdot \nabla p.$$  

Here $R = R_0 + r \cos \theta$, $R_0$: major radius, $r = a$: minor radius, and are the poloidal flux and a stream function, respectively. And

$$\mathbf{B} = B_0 \hat{\phi} + \nabla \psi \times \hat{\phi}, \quad \mathbf{v} = \nabla U \times \hat{\phi}, \quad j_\phi = -\nabla^2 A_\phi, \quad \psi = A_\phi + \psi_v(r, \theta), \quad \kappa_r = -\nabla \Omega, \quad \Omega = \frac{r}{R_0} + \Omega_v(r, \theta).$$

$\psi_v$, and $\Omega_v$ are the averaged vacuum poloidal flux and the vacuum magnetic curvature potential, respectively, which are zero for tokamaks, and non-zero for heliotron.

The potential energy based on reduced MHD equation is as the following:

$$\delta W(\xi, \xi) = \frac{1}{2} \int r \left[ (Q_\perp)^2 - 2(\xi_\perp \cdot \nabla p)(\kappa_r \cdot \xi_\perp) - j_\phi (\xi_\perp \times \hat{\phi}) \cdot Q_\perp \right]$$

The terms with the compressional alfven wave and the magnetic sound wave disappear. Here $\xi_\perp = \nabla U \times \hat{\phi}$, $Q_\perp = \nabla (\mathbf{B} \cdot \nabla U) \times \hat{\phi}$. 

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In order to linearize Eqs.(1), we assume \( U = \tilde{U}(r, \theta, \varphi), \quad p = p_0 + \tilde{p}(r, \theta, \varphi), \quad A_\varphi = A_0 + \tilde{A}(r, \theta, \varphi) \) and
\[
\{ \tilde{U}, \tilde{p}, \tilde{A} \} = \{ \hat{U}, \hat{p}, \hat{A} \} \exp[i(m\theta - n\varphi) - i\omega t].
\]

The following eigenmode equation for \( U \) is derived in the cylinder geometry,
\[
\omega^2 \left( \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} - k_\theta^2 \right) \hat{U} = \omega k_{||} \left( \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} - k_\theta^2 \right) \left( \frac{k_{||} \hat{U}}{\omega} \right) + k_\theta^2 p_0' \Omega' \hat{U}
\]
where \( k_{||} = m't - n, \quad k_\theta = m/r \) and primes denote the derivative with respect to \( r \).

In order to analyze the radially localized mode in the neighborhood of a resonant surface, the following relations are used; \( x = r - r_0, \quad k_{||} = k_{||}'x, \quad k_{||}' = m't |_{r = r_0}. \)
\[
\frac{d^2}{dr^2} \hat{U} - \left\{ \frac{k_{\theta_0}^2 p_0' \Omega'}{k_{||}'^2} + k_{\theta_0}^2 \right\} \hat{U} = 0
\]

The solution of \( U \) is assumed as \( U \sim x^\nu \). \( U \) crosses 0 around \( x = 0 \) infinite times when
\[-k_{\theta_0}^2 p_0' \Omega'/k_{||}'^2 > 1/4.\]

According to Sydum, when \( U \) crosses 0 without boundaries, the system is always unstable. Then \( -\frac{p_0' \Omega'}{r^2} > \frac{1}{4} t^2 \) is a sufficient condition of the localized instabilities.

Mercier criterion corresponds to the extended Sydum criterion to the toroidal system.
Generally speaking, the global mode is stable even that the sydum’s criterion is unstable. Why the global mode is stable there nevertheless sydum’s criterion is a sufficient condition of the instability?

The mode width of the interchange mode becomes narrower as the growth rate decreases. Usually the stability analysis is calculated with finite mesh size. Then the only unstable modes with finite size and finite growth rate can be analyzed. The sydum’s criterion corresponds to the instability condition for the limit of the radially localized mode.

Caution!!
The stability limit of the global unstable mode depends on the calculation mesh size especially in the interchange mode.
Ideal Interchange mode IV ---Example of the mode structure and growth rate of the interchange mode---

Because $w$ is imaginary, the difference of the phase between $U$ and $p$ is $\pi/2$. 

$$\tilde{p} = k_0 p_0 \tilde{U} / \omega$$
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磁力線の曲率と磁場強度の強弱

\[ \nabla \cdot B = 0 \] (磁力線は閉じるという性質)

より，磁力線の曲率が凸に曲がっている方向に，磁場強度は弱まる。

電流

磁力線

磁場弱
環状磁場プラズマでの交換型MHD不安定特性 II

環状磁場プラズマ閉じ込め装置では、ドーナッツの内、外で磁場の曲率が異なる
⇒ 悪い曲率での不安定性の発現
⇒ 磁気面上で不均一な不安定性 (バルーニングモード)
Pressure driven instabilities in torus plasmas

magnetic hill and well

\[ \text{When mag. flux averaged } \nabla B \text{ is negative (positive), it is magnetic hill (well) config.} \]
\[ \nabla B \text{ is locally negative (positive).} \Rightarrow \text{bad (good) curvature.} \]

1. Tokamaks

\[ \text{Mag. axis torus-outwardly shifted case} \]

Well/Hill depends on the relative location between 2 neighboring surfaces.

The averaging location of the mag. surf. is more torus inward as the minor radius increases.

\[ \Rightarrow \text{The averaged } B \text{ of mag. surf. increases as the larger minor radius.} \]

2. “Straight” heliotron

Averaged \( B \) of mag.surf. decreases as the larger minor radius.

\[ \Rightarrow \text{Mag. Well.} \]
Characteristics MHD equil. related to stability in LHD

Straight stellarator's mag. field is expressed as the followings.

\[ \mathbf{B} = B_h + B_0 z = \nabla \Phi + B_0 z \]

(helical field by 1 pair of helical coils + constant toridal field)

Mag. field by helical coils is expressed by a scalar potential \( \Phi \)

\[ \Phi = \sum_{l=4}^{l_{\text{max}}} \Phi_l(r) I_l(lM_r/R_0) \sin(l\theta - M_z/R_0) \]

In torus, \( M \); toridal pitch number, \( R_0 \); plasma major radius, \( \phi = z/R_0 \) (toroidal angle)

Here mag. flux due to helical coil, \( \psi_h \), is introduced.

\[ \psi_h = -\frac{1}{2B_0} \sum_{l,m} \Phi_l(r) \Phi^*_m(r) \frac{m}{r} I_l(lM_r/R_0) I_m(lM_r/R_0) \cos((l-m)\theta) \]

By using , averaged mag. field strength and rotational transform are expressed

\[ \Omega = \frac{B^2}{B_0^2} = \left| \frac{\nabla \Phi}{B_0} \right|^2 = -\frac{M}{lR_0B_0} \frac{1}{r} \frac{d}{dr} \left( r^2 \psi_h \right), \quad \text{and} \quad \iota_h = -\frac{R_0}{rB_0} \frac{d \psi_h}{dr}. \]

=> \[ \frac{d \Omega}{dr} = -\frac{M}{lR_0^2} \frac{1}{r^2} \frac{d}{dr} \left( r^4 \iota_h \right). \]

When \( d\iota_h/dr > 0 \),

\[ \Rightarrow \text{Magnetic hill structure.} \]

Characteristics MHD equil. related to stability in LHD II

Magnetic hill exists in the finite beta gradients region => MHD instabilities (interchange/pressure driven) would appear in high beta regime.

\[ \Lambda_p = 6.2, \, p \sim (1 - \rho^2)(1 - \rho^8) \]
Observation of the mode structure of the interchange mode in LHD

Profile of the radial displacement by ECE measurement and theoretical prediction

Evolution of the ECE perturbation

Toroidal Alfvén freq. ~$5.3 \times 10^6$ Hz @ 2.75T, $n_e = 10^{19} \text{m}^{-3}$, $R_0 = 3.6 \text{m}$

According to theoretical prediction, the growth rate is around $\sim 1.6 \times 10^4$ Hz (63 $\mu$s).

There is discrepancy between the prediction and observation in the growth rate.

The prediction of the ideal interchange mode is quite consistent with the observation on the mode structure.

Reduced MHD Equation

\[
\rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = \mathbf{j} \times \mathbf{B} - \nabla p,
\]

\[\frac{\partial p}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right)p = 0,\]

\[\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \quad \nabla \cdot \mathbf{B} = 0, \quad \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \ (\eta \mathbf{j}), \quad \text{Ohm's law  Maxwell eq.}\]

Here some quantities are linearized (separate the equilibrium part (suffix 0) and the perturbed part (suffix 1).).

\[\mathbf{v}(\mathbf{r}, t) = 0 + \mathbf{v}_1(\mathbf{r}, t), \quad \mathbf{j}(\mathbf{r}, t) = \mathbf{j}_0(\mathbf{r}) + \mathbf{j}_1(\mathbf{r}, t), |\mathbf{j}_1| \ll |\mathbf{j}_0|.\]

\[\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r}, t), |\mathbf{B}_1| \ll |\mathbf{B}_0|, \quad \mathbf{E}(\mathbf{r}, t) = 0 + \mathbf{E}_1(\mathbf{r}, t), \]

\[\rho(\mathbf{r}, t) = \rho_0(\mathbf{r}) + \rho_1(\mathbf{r}, t), \rho_1 \ll \rho_0, \quad p(\mathbf{r}, t) = p_0(\mathbf{r}) + p_1(\mathbf{r}, t), p_1 \ll p_0.\]

The 1st order momentum equations are as follows:

\[\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \mathbf{j}_1 \times \mathbf{B}_0 + \mathbf{j}_0 \times \mathbf{B}_1,\]

\[\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_1) = 0, \quad \frac{\partial p_1}{\partial t} = -\mathbf{v}_1 \cdot \nabla p_0 - \gamma \rho_0 \nabla \cdot \mathbf{v}_1,\]

\[\mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_0 = \mu_0 \mathbf{j}_1, \quad \frac{\partial \mathbf{B}_1}{\partial t} = -\nabla \times \mathbf{E}_1, \quad \nabla \times \mathbf{B}_1 = \mu_0 \mathbf{j}_1, \quad \nabla \cdot \mathbf{B}_1 = 0.\]
Reduced MHD Equation II

The 1st order momentum equations are as follows:

\[
\begin{align*}
\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} &= -\nabla p_1 + j_1 \times B_0 + j_0 \times B_1, \\
\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_1) &= 0, \\
\frac{\partial \mathbf{B}_1}{\partial t} &= \nabla \times (\mathbf{v}_1 \times B_0 - \eta \nabla \times B_1), \\
\nabla \cdot \mathbf{B}_1 &= 0.
\end{align*}
\]

Here \( \nabla \cdot \mathbf{v}_1 = 0 \) is assumed. \( \frac{\partial \rho_1}{\partial t} + \mathbf{v}_1 \cdot \nabla \rho_0 = 0. \)

Here \( \nabla \cdot (\mathbf{B} \times) \) for 1st eq., and by using \( \nabla \cdot \mathbf{v}_1 = 0 \) and \( \nabla \cdot \mathbf{B}_1 = 0. \)

\[
\begin{align*}
\mathbf{B} \cdot \frac{\partial v_1}{\partial t} &= \nabla \times \left( \mathbf{B} \cdot \nabla \right) j_1 \cdot \mathbf{B}_0 + \mathbf{B} \cdot \nabla B_0^2 \times \nabla p_1, \\
\frac{\partial \rho_1}{\partial t} + \mathbf{v}_1 \cdot \nabla \rho_0 &= 0, \\
\frac{\partial \mathbf{B}_1}{\partial t} &= (\mathbf{B}_0 \cdot \nabla) \mathbf{v}_1 - (\mathbf{v}_1 \cdot \nabla) \mathbf{B}_0 + \eta \nabla^2 \mathbf{B}_1.
\end{align*}
\]

\( \eta = 0, \mathbf{v}_1 = \nabla U \times \phi, \) and \( \mathbf{B}_1 = \nabla \psi_1 \times \phi \) are assumed. \( \mathbf{B}_0 = B_0 \phi + \nabla \psi_0 \times \phi, \ j_\phi = -\nabla_\perp^2 (\psi - \psi_0 (r, \Theta)), \ \psi = \psi_0 + \psi_1. \)

\[
\frac{\partial \psi}{\partial t} = \mathbf{B} \cdot \nabla U, \quad \left( \frac{\partial}{\partial t} + \mathbf{B} \cdot \nabla \right) \rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \nabla_\perp^2 U = -\mathbf{B} \cdot \nabla j_\phi - \phi \times \mathbf{k}_r \cdot \nabla \rho.
\]
Ideal Interchange mode I ---Reduced MHD equation---

When the high aspect approximation ($\varepsilon = a/R_0 \ll 1$) and the high beta ordering ($\beta \sim \varepsilon$) are applied to the full MHD equations, in a quasi toroidal coordinates, ($r$, $\theta$, $\phi$), the following reduced MHD equation is obtained. [$\xi = \text{grad} \ U \times z$ ($\xi = mU/r \omega$)]

$$\frac{\partial \psi}{\partial t} = B \cdot \nabla U, \quad \left( \frac{\partial}{\partial t} + B \cdot \nabla \right) p = 0, \quad \rho \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) \nabla \cdot U = -B \cdot \nabla j_\phi - \hat{\phi} \times \kappa_r \cdot \nabla p.$$

Here $R = R_0 + r \cos \theta$, $R_0$; major radius, $r = a$; minor radius. and are the poloidal flux and a stream function, respectively. And

$$B = B_0 \hat{\phi} + \nabla \psi \times \hat{\phi}, \quad v = \nabla U \times \hat{\phi}, \quad j_\phi = -\nabla \cdot A_\phi, \quad \psi = A_\phi + \psi_v(r, \theta), \quad \kappa_r = -\nabla \Omega, \quad \Omega = \frac{r}{R_0} + \Omega_v(r, \theta).$$

$\psi_v$ and $\Omega_v$ are the averaged vacuum poloidal flux and the vacuum magnetic curvature potential, respectively, which are zero for tokamaks, and non-zero for heliotron.

The potential energy based on reduced MHD equation is as the following;

$$\delta W(\xi, \xi) = \frac{1}{2} \int_{V_p} dV_p \left[ Q_\perp \cdot Q_\perp - 2(\xi_\perp \cdot \nabla p)(\kappa_r \cdot \xi_\perp) - j_\phi(\xi_\perp \times \hat{\phi}) \cdot Q_\perp \right].$$

The terms with the compressional alfven wave and the magnetic sound wave disappear. Here $\xi_\perp = \nabla \perp U \times \hat{\phi}$, $Q_\perp = \nabla \perp (B \cdot \nabla U) \times \hat{\phi}$.
In order to linearize Eqs.(1), we assume $U = \tilde{U}(r, \theta, \varphi)$, $p = p_0 + \tilde{p}(r, \theta, \varphi)$, $A_\varphi = A_0 + \tilde{A}(r, \theta, \varphi)$ and 
\[
\{\tilde{U}, \tilde{p}, \tilde{A}\} = \{\hat{U}, \hat{p}, \hat{A}\} \exp[i(m\theta - n\varphi) - i\omega t]
\]

The following eigenmode equation for $U$ is derived in the cylinder geometry,
\[
\omega^2 \left( \frac{1}{r} \frac{d}{dr} \frac{d}{dr} - k_\theta^2 \right) \hat{U} = \omega k_\parallel \left( \frac{1}{r} \frac{d}{dr} \frac{d}{dr} - k_\theta^2 \right) \left( \frac{k_\parallel \hat{U}}{\omega} \right) + k_\theta^2 p_0' \Omega' \hat{U}
\]
where $k_\parallel = mt - n$, $k_\theta = m/r$ and primes denote the derivative with respect to $r$.

In order to analyze the radially localized mode in the neighborhood of a resonant surface, the following relations are used; $x = r - r_0$, $k_\parallel = k_\parallel' x$, $k_\parallel' = m t' |_{r=r_0}$.
\[
\frac{d^2}{dr^2} \hat{U} - \left\{ \frac{k_\theta^2 p_0' \Omega'}{k_\parallel^{'2}} + k_\theta^2 \right\} \hat{U} = 0
\]

The solution of $U$ is assumed as $U \sim x^\gamma$. $U$ crosses 0 around $x=0$ infinite times when 
\[-k_\theta^2 p_0' \Omega'/k_\parallel^{'2} > 1/4.\]

According to Sydum, when $U$ crosses 0 without boundaries, the system is always unstable. Then 
\[-\frac{p_0' \Omega'}{r^2} > \frac{1}{4} t'^{2}\]

is a sufficient condition of the localized instabilities.

Mercier criterion corresponds to the extended Sydum criterion to the toroidal system.
When locally bad curvature exists in a magnetic flux surface, poloidally (and/or toroidally) localized pressure driven unstable mode appear there. => ballooning mode

Ex. Tokamak

Bad curvature

Unstable model appear localizedly here
Stability condition of ballooning instability

It is principally same with that in interchange mode. However local value of the curvature and magnetic shear are important in the bad curvature region.

\[- \frac{p_0' \Omega'}{r^2} > \frac{1}{4} t'^2 \implies -q^2 p_0' \tilde{\Omega}' > \frac{1}{4} \tilde{s}'^2 \left( \tilde{s} \equiv \frac{r}{q} \tilde{q}' = -\frac{r}{t} \tilde{t}' \right) \sim\]

means the value at bad curvature.

Even in tokamaks, \( \tilde{\Omega}' \) is positive.

\[\beta \text{ increases} \]

Shafranov shift occurs.

\[B_p \text{ increases where the magnetic fields becomes dense.} \]

Increment of \( B_p \) is larger where the the magnetic fields becomes more dense (the relative shift of the magnetic surface to the next magnetic surface is larger. In edge region, the relative shift is larger). \( \implies d\Delta B_p / dr > 0. \)
Stability condition of ballooning instability

Here \( \tilde{s} \equiv -q' - \alpha \), - \( \alpha \propto (\Delta q)' \) is the modification due to shafranov shift.

\[
(\Delta q)' = \frac{d}{dr} \left( \frac{r B_p}{\Delta B_{\rho} R} \right) = \frac{r B_p}{R} \frac{d}{dr} \left( \frac{1}{\Delta B_{\rho}} \right) < 0 \Rightarrow \alpha > 0.
\]

\[
(s - \alpha)^2 < k\alpha \Rightarrow s^2 - 2s\alpha + \alpha^2 - k\alpha < 0
\]

\[
\Rightarrow \alpha - \sqrt{k\alpha} < s < \alpha + \sqrt{k\alpha}
\]

According to more detail calculation, \( 0 < k < 1 \).

In tokamak, \( s > 0 \).
More exact treatment of stability analysis of ballooning mode

In order to exactly analyze the ballooning mode stability, we introduce the Eikonal approximation.

\[
A = F(r)\exp(iS), S(r, \theta, \phi) = -n[\phi - q(r)(\theta - \theta_0)] + [n\phi - m(r)(\theta - \theta_0)], \ n >> 1.
\]

When the Eikonal approximation is introduced, the potential energy is rewritten as follows.

\[
\delta W = \int \left[ (1 + \Lambda^2) \left( \frac{dF}{d\theta} \right)^2 - \alpha(\Lambda \sin \theta + \cos \theta)F^2 \right] d\theta, \text{ where } \Lambda = s - \alpha \sin \theta, s = r \frac{dq}{dr}, \ \alpha = -\frac{2\mu_0 Rq^2}{B^2} \frac{dp}{dr}.
\]

Minimization of the above potential energy leads to the following Euler equation,

\[
\frac{d}{d\theta} \left[ (1 + \Lambda^2) \left( \frac{dF}{d\theta} \right)^2 - \alpha(\Lambda \sin \theta + \cos \theta)F \right] = 0.
\]

There is no simple analytic solution of the Ballooning eq. Here just show a numerical calculation result. (If you want to know more detail analyzing procedure, please see refs.[1], [2],[3]).

According to refs.[2], the stability boundary is expressed as the following on $\delta$-\(\alpha\) diagram. Here $\delta$ is the amplitude of the magnetic well (averaged good curvature). It shows that in the averaged good curvature, the no ballooning mode unstable operation is possible.

Phenomena that heat and particles in the plasma edge region are oscillatingly exhausted in H-mode plasmas (Typically observed signal is the spick of $H\alpha$ ($D\alpha$) emission).

When ELMs occur, the impurities like He are efficiently exhausted. In H-mode operation without ELM, it is not easy to remove the impurities. Though confinement improvement of H-mode with ELM is less that that without ELM, it is a favorable phenomenon for controlling the impurities and the understanding of the mechanism is important.

H-mode;
A kind of the improved confinement mode. the significant reduction of particle loss in the edge region is observed ($H\alpha$ ($D\alpha$) signal is reduced). The steep gradients of the temperature and/or the density in the edge region appear.
ELM (Edge Localized Mode) II

There are some types.

Type I (giant) ELM
Type II (grassy) ELM

Frequency
Type I; 10~200Hz; proportional to the heating power across the separatrix.
Type II; higher than type I

5ms before type I ELM.
11ms after type I ELM.

A probable candidate of driving mechanism of ELM

Onset condition of type I ELM strongly depends on the triangularity.

The stability criterion also strongly depends on the triangularity.

=>

A probable candidate of the driving mechanism is ballooning mode.

---

A theoretical model of driving mechanism of ELM


-ELM cycle
Cycle I; Large ELMs (Type I ELM).
Cycle II; small ELMs (small Type I ELM or Type II).
Cycle III; low power, low density (Type III ELM).
Importance of the measurement of edge current profile for ELM study

Observation of the current profile in the edge region is important to identify the driving mechanism of ELM.

In DIII-D, Zeeman effect of the Li beam probe is used. Spatial resolution $\Delta R \approx 5\text{cm}$, 32 channel.

Application of the MSE measurement is not suitable for edge current profile measurement because the NBI beam attenuation is very small in the edge region.

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Physical picture of the current driven instabilities (kink modes)

Once cross-section of a mag. surf shrinks at a location, mag. field strength increases there, and mag. stress presses mag. surf. radially. => increases of mag. field strength.

Once column of mag. surface bends kink-like, mag. field strength increases in small curvature region and it decreases in large curv. region. Mag. stress in small curv. region is larger than that in large curv. region, and the column is pressed to bend more. => mag. field strength increases in small curvature region and it decreases in large curv. region.

Once plasma moves, it continues to move. => Unstable
Under presence of longitudinal field, additional force is affected as to suppress line bending. => Stabilizing effect

\[ B_\theta = \frac{I}{2\pi r} \]
\[ p + \frac{B^2}{2\mu_0} = \text{const.} \]
Here some quantities are linearlized (separate the equilibrium part (suffix 0) and the perturbed part (suffix 1)).

\[ \rho \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) v = j \times B - \nabla p, \]

Starting from ideal MHD equations

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \]

\[ \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) \left( \frac{p}{\rho^\gamma} \right) = 0, \]

\[ E + v \times B = 0 \quad (\eta \mathbf{j}), \]

\[ \nabla \times B = \mu_0 j, \quad \nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \cdot B = 0. \]

The 1st order momentum equations are as follows:

\[ \rho \frac{\partial v_1}{\partial t} = -\nabla p_1 + j_1 \times B_0 + j_0 \times B_1, \]

\[ \frac{\partial p_1}{\partial t} = -v_1 \cdot \nabla p_0 - \gamma p_0 \nabla \cdot v_1, \quad E_1 + v_1 \times B_0 = 0, \]

\[ \nabla \times B_1 = \mu_0 j_1, \quad \nabla \times E_1 = -\frac{\partial B_1}{\partial t}, \quad \nabla \cdot B_1 = 0. \]
Linearized MHD Eq. II

Summarizing the above equations,

\[ \rho_0 \frac{\partial^2 v}{\partial t^2} = -\nabla\{v_1 \cdot \nabla p_0 + \gamma p \nabla \cdot v_1\} + j_0 \times \{\nabla \times (v_1 \times B_0)\} + \frac{1}{\mu_0} \left[ \nabla \times \{\nabla \times (v_1 \times B_0)\}\right] \times B_0. \]

When \( v_1 \) replaces a Lagrangian variable, \( \partial \xi / \partial t \),

\[ \rho_0 \frac{\partial^2 \xi}{\partial t^2} = F(\xi), \]

\[ F(\xi) = -\nabla\{\xi \cdot \nabla p_0 + \gamma p \nabla \cdot \xi\} + \frac{1}{\mu_0} (\nabla \times B_0) \times Q + \frac{1}{\mu_0} (\nabla \times Q) \times B_0. \]

where \( Q \equiv \nabla \times (\xi \times B_0) \).

In the linear stability analysis, the following expression of the time evolution of the perturbation is useful,

\[ \xi(r, t) = \tilde{\xi}(r) \exp(-i \omega t). \] If \( \omega \) is imaginary, the mode grows (unstable).

Then

\[ -\rho_0 \omega^2 \xi = F(\xi). \]

Here it should be noticed that \( F \) is a self-adjoint operator, \( \int dV x \cdot F(y) = \int dV y \cdot F(x) \).

\[ \frac{1}{2} \rho_0 \omega^2 \int dV \tilde{\xi}^* \tilde{\xi} \xi = -\frac{1}{2} \int dV \tilde{\xi}^* F(\xi), \quad \frac{1}{2} \rho_0 \omega^2 \int dV \tilde{\xi} \tilde{\xi}^* = -\frac{1}{2} \int dV \xi F(\xi^*). \]

\[ \Rightarrow \frac{1}{2} \rho_0 \omega^2 \int dV \tilde{\xi}^* \xi \xi = \frac{1}{2} \rho_0 \omega^2 \int dV \tilde{\xi} \tilde{\xi}^* \Rightarrow \omega^2 = \tilde{\omega}^2. \]

Here \( * \) denotes a complex conjugate.
Linearized MHD Eq. III

\[
\frac{1}{2} \rho_0 \omega^2 \int dV \xi^* \xi = -\frac{1}{2} \int dV \xi^* F(\xi) \Rightarrow K \equiv \frac{1}{2} \rho_0 \int dV \xi^* \xi, \quad \delta W \equiv -\frac{1}{2} \int dV \xi^* F(\xi).
\]

\[\Rightarrow \omega^2 K = \delta W.\]

\[\Rightarrow \omega^2 = \frac{\delta W}{K}.\]

Because $K$ is positive, the sign of $\delta W$ determines the stability of the system. $\delta W > 0 \Rightarrow$ stable. $\delta W < 0 \Rightarrow$ unstable.

Here $K$ and $\delta W$ correspond to the kinetic energy and the potential energy. After some calculations, $\delta W$ is rewritten as

\[
\delta W = \frac{1}{2} \int dV_{\text{plasma}} \left[ \rho_0 (\nabla \cdot \xi)^2 + (\xi \cdot \nabla \rho_0)(\nabla \cdot \xi) - \frac{|Q|^2}{\mu_0} j_0 (Q \times \xi) - \xi \cdot (j_0 \times \mathbf{B}_i) \right].
\]

Change of the internal energy of plasma without magnetic energy

Change of the magnetic energy

Work against the unbalanced magnetic force
Linearized MHD Eq. IV

\[ \delta W = \frac{1}{2} \int dV_{\text{plasma}} \left[ \frac{|Q_\perp|^2}{\mu_0} + \frac{B_0^2}{\mu_0} \left| \nabla \cdot \xi_\perp + 2\xi_\perp \cdot \mathbf{k} \right|^2 + \nabla p_0 (\nabla \cdot \xi_\perp)^2 - 2(\xi_\perp \cdot \nabla p_0)(\xi_\perp \cdot \mathbf{k}) - j_n \cdot (Q_\perp \times \xi_\perp) \right] \]

- shear alfven wave
  (line bending term)
- sound wave of plasma
- compressional alfven wave
- current driven destabilizing term
- stabilizing term

\[ (k_\perp a)^2(k_\parallel a)^2 \]

When the mode is localized \( k_\perp a >> 1 \), pressure driven mode is dominant.

Current driven mode;
The global mode is more easily unstable than the localized mode. \( k_\perp a \sim 1 \)
External kink I

\[ \delta W = \frac{1}{2} \int_{\text{plasma}} \left[ \mathcal{P}_0 (\nabla \cdot \xi)^2 + (\xi \cdot \nabla \mathcal{P}_0) (\nabla \cdot \xi) + \frac{|Q|^2}{\mu_0} - j_0 (Q \times \xi) \right] + \int_{\text{vacuum}} \left( \frac{B_v^2}{\mu_0} \right). \]

\[ \beta \sim \epsilon^2, \quad \frac{\partial}{\partial r} \sim \frac{1}{r} \frac{\partial}{\partial \theta} \gg \frac{1}{R} \frac{\partial}{\partial \phi}, \quad \nabla \cdot \xi = 0 \quad \text{as the minimizing perturbation.} \]

\[ \delta W = \pi R \int_{\rho}^{a} \left( \frac{|Q|^2}{\mu_0} - j_{z0} (Q_r \xi_\theta - Q_\theta \xi_r) \right) d\theta dr + \pi R \int_{a}^{b} \left( \frac{B_v^2}{\mu_0} \right) d\theta dr. \]

where \(|Q|^2 = Q_\rho^2 + Q_\theta^2\), \(a\) is the plasma radius and \(b\) the radius of the perfect conducting wall. The perturbations are Fourier analyzed in the form \(\exp[i(m\theta - n\phi)]\), and becomes

\[ \xi_\theta = -\frac{i}{m} \frac{d}{dr} (r \xi_r). \]

Here using \(Q=\nabla \times (\xi x B_0)\), then

\[ Q_r = -\frac{imb}{R} \left( \frac{n}{m} - \frac{1}{q} \right) \xi_r, \quad Q_\theta = \frac{B_e}{R} \frac{d}{dr} \left( \frac{n}{m} - \frac{1}{q} \right) \xi_r. \]

where \(q (=rB_\phi/RB_\theta)\) is the safety factor. And \(\mu_0 j_{z0} \xi_\theta = \nabla x B_0 = (1/r)[d/dr(rB_\theta)]\).

\[ \delta W_p = \frac{\pi^2 B_\phi^2}{\mu_0 R} \int_{\rho}^{a} r dr \left( \frac{n}{m} - \frac{1}{q} \right)^2 \xi_r^2 + \left[ \frac{d}{dr} \left( \frac{n}{m} - \frac{1}{q} \right) r \xi_r \right]^2 + \frac{1}{r} \frac{d}{dr} \left( \frac{n}{m} - \frac{1}{q} \right) r \xi_r + \frac{d}{dr} \left( \frac{n}{m} - \frac{1}{q} \right) r \xi_r \]

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After the integration by parts in the term involving $\xi_r d\xi_r/dr$, 

$$\delta W_p = \frac{\pi^2 B^2}{\mu_0 R} \int_0^a rdr \left\{ \left( r \frac{d\xi_r}{dr} \right)^2 + \left( m^2 - 1 \right) \xi_r^2 \right\} \left( \frac{n}{m} - \frac{1}{q} \right) + \left[ \frac{2}{q_a} \left( \frac{n}{m} - \frac{1}{q_a} \right) + \left( \frac{n}{m} - \frac{1}{q_a} \right)^2 \right] a^2 \xi_{ra}^2$$

where the subscript $a$ denotes the value at $r=a$.

Next we consider the vacuum contribution to $\delta W$. In the vacuum, the perturbed magnetic field is expressed by a flux function $\Psi$ as $B_{v_r}=-\left(1/r \right) \partial \Psi / \partial \theta$ and $B_{v \theta} = \partial \Psi / \partial r$. Since

$$B_v^2 = B_{v_{r1}} + B_{v_{\theta1}} = \frac{m^2}{r^2} \Psi + \left( \frac{d\Psi}{dr} \right)^2,$$

the $\delta W_v$ is written as

$$\delta W_v = \frac{\pi^2 R}{\mu_0} \int_a^b r dr B_v^2 = \frac{\pi^2 R}{\mu_0} \left\{ \int_a^b r dr \left[ \frac{m^2}{r^2} \Psi - \frac{\Psi}{r} \frac{d}{dr} \left( r \frac{d\Psi}{dr} \right) \right] + \left( r \frac{d\Psi}{dr} \right) \bigg|_a^b \right\}.$$

Here it is noted that $\Psi$ satisfies the following Laplace’s equation, $\frac{1}{r} \frac{d}{dr} \left( r \frac{d\Psi}{dr} \right) - \frac{m^2}{r^2} \Psi = 0$.

Then

$$\delta W_v = \frac{\pi^2 R}{\mu_0} \left( r \frac{d\Psi}{dr} \right) \bigg|_a^b.$$

Here we assume the solution of the Laplace’s equation is written as $\Psi=\alpha r^m - \beta r^{-m}$.

For the conducting wall at $r=b$, $B_r(b)=0 \Rightarrow \Psi=0$ at $r=b$.

For the plasma surface $r=a$, $B_{v_{r1}}= -im \Psi_a/a = Q_r(a) = -i(mB_\phi/R)(n/m-1/q_a) \xi_r \Rightarrow 
\Psi_a = B_{\theta a}(nq_a/m-1) \xi_{ra}$. 

External kink II
The solution of the $\Psi$ is given as
$$\Psi = B_{\theta i} \left( \frac{nq_a}{m} - 1 \right) \frac{(r/b)^m - (b/r)^m}{(a/b)^m - (b/a)^m} \xi_{ra}.$$ 

Then the vacuum contribution $\delta W_v$ is expressed as
$$\delta W_v = \frac{\pi^2 R}{\mu_0} m \lambda \left( \frac{n}{m} - \frac{1}{q_a} \right) a^2 \xi_{ra}^2,$$
where $\lambda \equiv \frac{1 + (a/b)^{2m}}{1 - (a/b)^{2m}}$.

The above equation is added to the plasma contribution $\delta W_p$, 

$$\delta W = \frac{\pi^2 B_0^2}{\mu_0 R} \int_0^a r dr \left( \left[ \left( r \frac{d \xi_r}{dr} \right)^2 + (m^2 - 1) \xi_r^2 \right] \left( \frac{n}{m} - \frac{1}{q_a} \right)^2 \right) + \left[ \frac{2}{q_a} \left( \frac{n}{m} - \frac{1}{q_a} \right) + (1 + m \lambda) \left( \frac{n}{m} - \frac{1}{q_a} \right)^2 \right] a^2 \xi_{ra}^2.$$ 

When $\xi_{ra} = 0$, $\delta W \geq 0$. $\Rightarrow$ stable or marginal stable.

The integral contribution (inside of the plasma) is always positive or zero. When we consider the case that the $q_a > 0$, $m > 0$, $-\infty < n < \infty$, the necessary condition for instability is 

$$\left[ \frac{2}{q_a} + (1 + m \lambda) \left( \frac{n}{m} - \frac{1}{q_a} \right) \right] \left( \frac{n}{m} - \frac{1}{q_a} \right) < 0 \Rightarrow \frac{m m \lambda - 1}{n m \lambda + 1} < q_a < \frac{m}{n},$$

$$\Rightarrow \frac{m m - 1}{n m + 1} < q_a < \frac{m}{n} \quad (m = 2, n = 1 \Rightarrow 2/3 < q_a < 2, m = 3, n = 1 \Rightarrow 3/2 < q_a < 3) \quad @ \lambda = 1.$$
When \( m=1 \), the minimizing eigenfunction in the plasma is given by 
\[
\xi_r(r) = \xi_{ra} = \text{constant. Then}
\]

\[
\delta W = \left[ 2n \left( n - \frac{1}{q_a} \right) \right] a^2 \xi_{ra}^2 \Rightarrow q_a > 1 \geq \frac{1}{n} \text{ (the condition of stabilization, Kruskal-Shafranov condition.)}
\]

When \( m \geq 2 \), the trial function which minimizes the integral term (plasma term) should satisfy the following equation (Euler-Lagrange eq.)

\[
\frac{d}{dr} \left[ r^3 \left( \frac{n}{m} \right)^2 \frac{d \xi_r}{dr} \right] - r \left( m^2 - 1 \right) \left( \frac{n}{m} \right)^2 \xi_r = 0.
\]

Assuming the above eq. can be solved for \( \xi_r \), one can rewrite \( \delta W \) as follows:

\[
\delta W = \left( n - \frac{1}{q_a} \right) \left[ \left( m + \frac{a \xi_{ra}'}{\xi_{ra}} \right) \left( n - \frac{1}{q_a} \right) + \left( n + \frac{1}{q_a} \right) \right] a^2 \xi_{ra}^2.
\]

To evaluate \( \delta W \) the Euler-Lagrange eq. should be solved and \( a \xi_r / \xi_r \) be calculated. Assuming the mode structure is localized near the plasma boundary and after some calculation, we obtain the unstable criterion taking the current profile into account.

\[
\frac{1}{n} \left( m - \frac{J_a}{\langle J \rangle} \right) < q_a < \frac{m}{n}, \quad \text{for } J_a \neq 0, \quad \text{and} \quad \frac{1}{n} \left( m - \exp \left( \frac{2m \langle J \rangle}{a J_a} \right) \frac{J_a}{\langle J \rangle} \right) < q_a < \frac{m}{n}, \quad \text{for } J_a = 0.
\]
How fast the current gradient must vanish is difficult estimate by the analytical methods.

The right figure is the numerical results of the unstable region for external kinks for a current profile
\[ j(r) = j_0 (1 - \rho^2)^\nu. \]
\[ \rho = r/a. \]
\[ q = q_a \rho [1 - (1 - \rho)^{\nu + 1}]^{-1}. \Rightarrow q_a/q_0 = \nu + 1. \]

When \( \nu > 2.5 \), all \( m \geq 2 \) modes are stable for any \( q_a \).
When \( 1 < \nu < 2.5 \), the stability window of \( q_a \) exists.

J.A. Wesson; Nucl. Fus. 18, 87 (1978).
External kink VI ---Observation of the mode structure of the external kink mode in TFTR---

Mode structure is broad.

\[ \tilde{\xi}_r = \frac{T_e}{(-\nabla T_e)} \]

Profile of the radial displacement by ECE measurement and theoretical prediction

\[ \omega_A = 1.7 \times 10^4 \sim 5 \times 10^4 \text{ Hz} \]

\[ n_e = 1 \times 10^{20} \sim 1 \times 10^{19} \text{ m}^{-3} \]

\[ \delta B (\text{G}) = 3 \]

\[ \gamma_{\text{grow}} / \omega_A = 0.5 \sim 0.2 \]

The prediction of the external kink mode is quite consistent with the observation on the mode structure.

Tearing mode --Physical picture--

\[
j_z(x) = \begin{cases} 
  j_{z0} & -a < x < a \\ 
  0 & |x| > a 
\end{cases}
\]

\[
B_y(x) = \begin{cases} 
  B_{y0}x & -a < x < a \\ 
  -B_{y0}a & x < -a \\ 
  B_{y0}a & x > a 
\end{cases}
\]

If \( B_y = 0 \) in \(|x| < \delta\), \( \int \left(B_y^2/2\right) dV \) is reduced.

Without resistivity

With resistivity

Tearing mode I

With resistivity

\[ \Rightarrow \text{change of topology of magnetic field structure} \]

\[ [\partial_t \mathbf{B} = -\nabla \times (\mathbf{v} \times \mathbf{B} - \eta \mathbf{j})] \Rightarrow i \omega B_{y1} = -k \gamma B_{y0} u_{x1} \eta \frac{\partial B_{x1}^2}{\partial x^2} \]

If \( \eta = 0 \), \( B_{x1} = 0 \) at \( x = 0 \) because \( B_{y0} = 0 \). If \( \eta \neq 0 \), \( B_{x1} \neq 0 \) at \( x = 0 \)

\[ \mu_0 j_{z1} = B_{y1}|_{0+} - B_{y1}|_{0-}, \quad \frac{\partial B_{x1}}{\partial x} + ik B_{y1} = 0, \quad \Delta B_{x1} \equiv \frac{\partial B_{x1}}{\partial x} |_{0+} - \frac{\partial B_{x1}}{\partial x} |_{0-} \]

\[ \Rightarrow \mu_0 j_{z1} = -i \Delta B_{x1} / k_y \]

When \( j_{z1} \) exists, tearing mode is unstable.
Then \( \Delta > 0 \Rightarrow \text{unstable.} \)
Tearing mode II

From linearized momentum eq.

\[ \frac{\partial}{\partial x} \left[ B_{y0}^2 \frac{\partial}{\partial x} \left( \frac{B_x}{B_{y0}} \right) \right] - k_y^2 B_{y0} B_{x1} = 0, \text{ for "outer - regions".} \]

\[ - \omega \rho_0 \mu_0 \frac{\partial^2 u_{x1}}{\partial x^2} = \frac{k_y B_{y0}}{i \eta} (\omega B_{x1} + k_y B_{y0} u_{x1}), \text{ for "resistive layer (} |x| < \delta \equiv \frac{\eta}{k_y B_{y0} u_{x1}} \frac{\partial^2 B_{x1}}{\partial x^2} \text{)".} \]

\[ \Delta^e \text{ is determined.} \]

Relationship between \( \Delta^e \) and \( \gamma (-i\omega) \) is determined.

\[ \gamma = \left( \frac{\tau_R}{\tau_A} \right)^{0.4} \frac{0.55(\Delta a)^{4/5}}{\tau_R}. \]

Magnetic Reynolds number \( \sim 10^8 \) in JT-60.

\( \Rightarrow \) very large

\( \tau_R; \) magnetic diffusive time \( \sim \text{min} \) is very long,
but \( \gamma \) of tearing mode is fairly large.

(\( \tau_A \sim 10^{-6} \text{s} \))
Tearing mode III

When we solve the momentum eq. in the outer regions

\[
\frac{\partial}{\partial x} \left[ B_{y0}^2 \frac{\partial}{\partial x} \left( \frac{B_{x1}}{B_{y0}} \right) \right] - k_y^2 B_{y0} B_{x1} = 0,
\]

under the following configurations,

\[
B_y(x) = \begin{cases} 
B'_{y0} x & -a < x < a \\
-B'_{y0} a & x < -a \\
B'_{y0} a & x > a 
\end{cases}
\]

we obtain the solution,

\[
\Delta a = \frac{2k_y \left[ \exp(-2k_ya) - 2k_ya + 1 \right]}{\exp(-2k_ya) + 2k_ya - 1}.
\]

Generally speaking, when \( k_y \) is small, the tearing mode easily becomes unstable.

\[ \Rightarrow \] For the large \( k_y \) mode, the line-bending stabilization easily becomes large.
The existence of bootstrap current is indispensability to obtain the advanced tokamak plasmas because it helps to reduce the external current drive and to make the negative shear configuration.

However, when magnetic island exists, the bootstrap current affects the MHD stability through the neoclassical tearing mode.
The model eq. of the evolution of the island width (Modified Rutherford eq.)

\[
\frac{\tau_s}{\tau} \frac{d}{dr} \left( \frac{w}{r_s} \right) = \delta'(w) r_s - k_0 e_s^2 \beta_{ps} \frac{L_0}{L_p} \left( 1 - q_s^{-2} \right) \frac{1}{w} + k_1 r_s \sqrt{\epsilon_s} \beta_{ps} \frac{L_n}{L_0} \frac{w}{w_0^2 + w_0^2} - k_2 r_s \rho_{ps}^2 \beta_{ps} \left( \frac{L_n}{L_0} \right)^2 g(\nu, \varepsilon) \frac{1}{w^3} - k_3 \frac{16 \mu_0 L_n I_{aux, f_{aux}}}{\pi B_0} \frac{1}{w^3}
\]

- The seed island due to tearing mode
- Destabilizing term due to BS current
- Stabilizing term due to external current
- Stabilizing term due to mag. well
- Stabilizing term due to polarization current

Example of observation of the neoclassical tearing mode

**Observation;**
After \( b \) have exceeded 1.2%, it and \( S_n \) saturate, and just before the saturation, \( m/n=3/2 \) fluctuation starts increasing.

**How is it identified?**
Comparing measured \( m/n = 3/2 \) island width with the theory. Mod. Rutherford model works well to explain the observation.


Neutron production rate due to DD reaction in TFTR

Comparison of the measured \( m/n = 3/2 \) island width (labeled 'a') with the neoclassical tearing mode theory (curve 'b' uses the time dependent parameters and 'c' uses fixed parameters)
Disruption I

A dramatic event in which the plasma confinement is suddenly destroyed. In tokamak operations, it is often observed.

=> As a main cause, the instability related with the current driven mode and the subsequent destruction of the magnetic surfaces.

Typical sequence
1. Precursor in mag. probe signals are observed.
2. Temperature suddenly decreases.
2’. The equilibrium is destroyed less than once. And it is considered to be reconstructed.
3. Density and net toroidal current decrease slower than temperature does.

*Decay time of density is mainly determined by particle confinement of the cold plasma, and that of net toroidal current is by L/R time.*
Disruption II

The radial over-rapping of tearing modes, a external kink and the positional instability are considered as main causes.

An example of the growth of MHD instabilities due to tearing modes in a disruption.


\[
\mathbf{B} \times \nabla = -\dot{\mathbf{E}}
\]

=>

\( B_r \) component monotonically increases.

\( q_0 \) should be larger than 1. and the smaller toroidal current flows as the minor radius increases.

=> The central current profile is flattened. And the current gradient increases near \( q=2 \) surface.

=> The tearing mode becomes easily unstable.
Disruption III

When over-rapping of the island due to tearing modes occur, magnetic surfaces are destroyed, and the confinement will be lost.

When a external kink and the positional instability happen, the plasma losses rapidly due to a touch of the wall. => Rapid decay of temperature
Sawtooth oscillation

Typical behavior
At first, the inner signal gradually increases and the outer one decreases. And at a time, the inner signal suddenly drops, and at the same time the outer signal increases. It is repeated.

=> The inner temperature (and/or density) suddenly drops, and the outer one increases.

The collapse is due to the instability with $m/n=1/1$ structure. It usually starts when a $q=1$ surface appear in plasma region.

ref. ASDEX team, Nucl. Fusion 29, 891 (1989).


Radial profile of fluctuation of SX
A theoretical model for sawtooth oscillation

Kadomtsev’s model
m/n=1/1 instability displaces the center region of plasma. The X point is created, the mag. field line breaks and reconnected where the magnetic flux is crowded. In intermediate state, a cooler core island is surrounding the displaced hot circular core.

According to Kadomtsev’s model, the collapse time is \( \tau_R (\tau_A/\tau_R)^{0.5} \)

ref J.Wesson; Chap.7.6 in “Tokamaks --2nd edit.--” Clarendon press (1997).
m=1 internal current driven modes

The leading order of $\delta W$ respect with $r/R$ is zero. When we consider higher order, the $\delta W$ is expressed as the followings.

$$
\delta W = \left(1 - \frac{1}{n^2}\right) \delta W_c + 2\pi^2 R \xi_0^2 \frac{B_\phi^2}{\mu_0} \left(\frac{r_1}{R}\right)^4 \delta W_T
$$

Cylinder effect; in tokamaks, $q>0.5$. Then this term always positive.

Toroidal effect; When $\beta_{p1}>0.3$, the ideal internal kink mode becomes unstable. However, the growth rate is small because $\delta W$ is small. It should be noted that ideal internal kink mode is unstable just in finite pressure plasma.

When we consider the resistive effect on the m/n=1/1 mode. That is unstable even without plasma pressure.

$$
\gamma = \left(\frac{r}{R} \frac{aq}{q}\right)^{2/3} \left(\frac{\tau_R}{\tau_A}\right)^{2/3}
$$

There are stronger effect of the resistivity. When the resistivity becomes large, it more easily destabilized than $m=2$ tearing mode.

Comparison between exp. observation and Kadomatsev’s mode

ref J. Wesson; Chap. 7.6 in “Tokamaks --2nd edit.--” Clarendon press (1997).

Some discrepancies exist between observation and the model; **still open question!**

(1) According to a observation of $q_0$, $q_0$ remains well below 1, which is conflict with the model which predicts full reconnection.
(2) According to the model, the collapse time is larger as the device becomes larger because of the increase of $\tau_A/\tau_R$, which is conflict with the experimental results that the collapse time hardly depends on the device size.
(3) According to a observation of fluc. contour, it looks that cold bubble invade the core, hot island is surrounding it, which is conflict with the model which predicts a cooler core island is surrounding the displaced hot circular core.

It looks that cold bubble invade the core, hot island is surrounding it.

Time evolution of observed $q_0$

ECE tomography result in JET
Index of lecture

1. Introduction of MHD  Jan. 5

2. MHD equilibrium  Jan. 5

3. Pressure driven MHD instabilities  Jan. 12
   Interchange mode
   Ballooning mode  Jan. 19

4. Current driven MHD instabilities  Jan. 19

5. Hot topics of MHD equilibrium and instability  Jan. 27
On next phase of this class,

Start from Feb. 1 (Tue.) 13:30~
At meeting room on 7th floor.
Teacher; Prof. Miyazawa
Mode coupling through configuration effects through Non-linear process
Reduced MHD Equation

\[ \rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = \mathbf{j} \times \mathbf{B} - \nabla p, \quad \text{Motion eq.} \]

\[ \frac{\partial p}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) p = 0, \quad \text{Eq. of continuity and state for } \gamma = 0 \]

\[ \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \quad \nabla \cdot \mathbf{B} = 0, \quad \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \quad (\eta \mathbf{j}), \quad \text{Ohm's low Maxwell eq.} \]

Here some quantities are linearized (separate the equilibrium part (suffix 0) and the perturbed part (suffix 1).).

\[ \mathbf{v}(\mathbf{r}, t) = 0 + \mathbf{v}_1(\mathbf{r}, t), \quad \mathbf{j}(\mathbf{r}, t) = \mathbf{j}_0(\mathbf{r}) + \mathbf{j}_1(\mathbf{r}, t), \quad |\mathbf{j}_1| << |\mathbf{j}_0|. \]

\[ \mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r}, t), \quad |\mathbf{B}_1| << |\mathbf{B}_0|, \quad \mathbf{E}(\mathbf{r}, t) = 0 + \mathbf{E}_1(\mathbf{r}, t), \]

\[ \rho(\mathbf{r}, t) = \rho_0(\mathbf{r}) + \rho_1(\mathbf{r}, t), \quad \rho_1 << \rho_0, \quad p(\mathbf{r}, t) = p_0(\mathbf{r}) + p_1(\mathbf{r}, t), \quad p_1 << p_0. \]

The 1st order momentum equations are as follows:

\[ \rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \mathbf{j}_1 \times \mathbf{B}_0 + j_0 \times \mathbf{B}_1, \]

\[ \frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_1) = 0, \quad \frac{\partial p_1}{\partial t} = -\mathbf{v}_1 \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \mathbf{v}_1, \]

\[ \mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_0 = \mu \mathbf{j}_1, \]

\[ \frac{\partial \mathbf{B}_1}{\partial t} = -\nabla \times \mathbf{E}_1, \quad \nabla \times \mathbf{B}_1 = \mu \mathbf{j}_1, \quad \nabla \cdot \mathbf{B}_1 = 0. \]
Reduced MHD Equation II

The 1st order momentum equations are as follows:

\[ \rho_0 \frac{\partial v_1}{\partial t} = -\nabla p_1 + j_1 \times B_0' + j_0 \times B_1', \]

\[ \frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 v_1) = 0, \quad \frac{\partial p_1}{\partial t} + v_1 \cdot \nabla p_0 = 0, \]

\[ \frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (v_1 \times B_0' - \eta \nabla \times B_1'), \quad \nabla \cdot \mathbf{B}_1 = 0. \]

Here \( \nabla \cdot \mathbf{v}_1 = 0 \) is assumed. \( \frac{\partial \rho_1}{\partial t} + v_1 \cdot \nabla \rho_0 = 0. \)

\[ \text{incompressible approximation} \]

Here \( \nabla \cdot (\mathbf{B} \times) \) for 1st eq., and by using \( \nabla \cdot \mathbf{v}_1 = 0 \) and \( \nabla \cdot \mathbf{B}_1 = 0. \)

\[ \mathbf{B} \cdot \rho_0 \nabla \times \frac{\partial v_1}{\partial t} = \left( \mathbf{B} \cdot \nabla \right) j_1 \cdot B_0' + \mathbf{B} \cdot (B_0' \nabla p_1), \]

\[ \frac{\partial p_1}{\partial t} + v_1 \cdot \nabla p_0 = 0. \]

\[ \frac{\partial \mathbf{B}}{\partial t} = (B_0 \cdot \nabla) v_1 - (v_1 \cdot \nabla) B_0' + \eta \nabla^2 \mathbf{B}_1. \]

vorticity eq. \( \nabla \times \mathbf{v}_1; \) vortecity

energy (density) conservation eq.

Faraday’s law + Ampere, Ohms low

\[ \mathbf{v}_1 = \nabla U \times \Phi, \text{ and } \mathbf{B}_1 = \nabla \psi \times \Phi \text{ are assumed. } B_0' = B_0 \Phi + \nabla \psi \times \Phi, \quad j_{\varphi 1} = -\nabla \times^2 \psi. \]

\[ \rho_0 \frac{\partial}{\partial t} \nabla_{\perp}^2 U = -B_0 \cdot \nabla \left( \nabla_{\perp}^2 \psi \right) - \Phi \times \kappa_r \cdot \nabla p_1, \quad \frac{\partial p_1}{\partial t} + v_1 \cdot \nabla p_0 = 0, \quad \frac{\partial \psi}{\partial t} = B_0 \cdot \nabla U + \eta \nabla_{\perp}^2 \psi. \]
Mode coupling I

\[
\rho_0 \frac{\partial}{\partial t} \nabla U \perp = -B_0 \cdot \nabla \left( \nabla U \right) - \phi \times k_r \cdot \nabla p_1, \\
\rho \frac{\partial p_1}{\partial t} + \mathbf{v}_1 \cdot \nabla p_0 = 0, \\
\frac{\partial \psi}{\partial t} = B_0 \cdot \nabla U + \eta \nabla U \perp.
\]

\[
\mathbf{v}_1 = \nabla U \times \phi, \\
j_{p1} = -\nabla U \perp.
\]

\[
\rho_0 \frac{\partial^2}{\partial t^2} \nabla U \perp = -B_0 \cdot \nabla \left( \frac{\partial \psi}{\partial \theta} \right). \\

\]

A kind of wave

Fourier-Laplace expansion; torus plasmas has 2 type of period.

\[
\{U, p, A\} = \sum_{m, n} \{\hat{U}_{mn}, \hat{p}_{mn}, \hat{A}_{mn}\} \exp[i (m \theta - n \phi) - i \omega t].
\]

\(\hat{U}_{mn}, \hat{p}_{mn}, \hat{A}_{mn}\) are function of \(\rho\) (radial variable; index of mag. surface)
Mode coupling II --through configuration effects--

\[ \rho_0 \frac{\partial^2}{\partial t^2} \nabla \cdot \nabla U - B_0^2 \cdot \nabla \left( \nabla \cdot \nabla U \right) \]

\[ \rho_0 \frac{\partial^2}{\partial t^2} \sum_{mn} X_{mn} \exp \left( i(m\theta - n\phi) - i\omega t \right) - B_0^2 \cdot \nabla \left( \sum_{mn} X_{mn} \exp \left( i(m\theta - n\phi) - i\omega t \right) \right) \]

When \( B_0 \) (equilibrium field) is homogeneous case, there is no mode coupling. => \( X_{mn} \exp(i(m\theta-n\phi)-i\omega t) \) is independent each other.

When \( B_0 \) (equilibrium field) is inhomogeneous case like tokamaks \([B_0 = B_0^*R_0/(R_0+r^*\cos\theta)]\), mode coupling happens. => mode coupling through equilibrium config.

=> When \( \omega \) is fixed, \( \sum X_{mn} \exp(i(m\theta-n\phi)-i\omega t) \) is determined.

Notes!
There are some means in word "mode".
When \( \omega \) is fixed, the space structure (combination set of m,n, \( X_{mn}(\rho) \)) is determined. The set including \( \omega \) means a mode.
A component of \( \sum X_{mn} \exp(i(m\theta-n\phi)-i\omega t) \) is sometimes called "mode".
Example of mode coupling through equil. configuration


external kink

Tokamaks ballooning mode
Reduced MHD Equation II'

The 1st order momentum equations are as follows:

\[
\frac{\rho_0}{\partial t} \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + j_i \times \mathbf{B}_0 + j_0 \times \mathbf{B}_1 + j_i \times \mathbf{B}_1,
\]
\[
\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_1 + \rho_1 \mathbf{v}_1) = 0, \quad \frac{\partial p_1}{\partial t} + \mathbf{v}_1 \cdot \nabla p_0 + \mathbf{v}_1 \cdot \nabla p_1 = 0,
\]
\[
\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0 + \mathbf{v}_1 \times \mathbf{B}_1 - \eta \nabla \times \mathbf{B}_1), \quad \nabla \cdot \mathbf{B}_1 = 0.
\]

\[
\frac{\rho_0}{\partial t} \nabla_\perp^2 \mathbf{U} = -\mathbf{B}_{0_1} \cdot \nabla \left( \nabla_\perp^2 (\psi_{h_0} + \psi) \right) - \mathbf{\hat{\phi}} \times \kappa \cdot \nabla p_1,
\]
\[
\frac{\partial p_1}{\partial t} + \mathbf{v}_1 \cdot \nabla (p_0 + p_1) = 0,
\]
\[
\frac{\partial \psi}{\partial t} = \mathbf{B}_{0_1} \cdot \nabla \mathbf{U} + \eta \nabla_\perp^2 \psi.
\]

\[
\rho_0 \frac{\partial^2}{\partial t^2} \nabla_\perp^2 \mathbf{U} = -\mathbf{B}_0 \cdot \nabla \left( \nabla_\perp^2 \frac{\partial \psi}{\partial t} \right) - \mathbf{B}_1 \cdot \nabla \left( \nabla_\perp^2 \frac{\partial \psi}{\partial t} \right), \quad \frac{\partial \psi}{\partial t} = \mathbf{B}_0 \cdot \nabla \mathbf{U} + \mathbf{B}_1 \cdot \nabla \mathbf{U}
\]

\[
\rho_0 \frac{\partial^2}{\partial t^2} \nabla_\perp^2 \mathbf{U} \sim -\mathbf{B}_0 \cdot \nabla \nabla \left( \nabla_\perp^2 \mathbf{U} \right) + \nabla_\perp^2 \mathbf{U} \times \nabla \nabla \left( \nabla_\perp^2 \mathbf{U} \right)
\]
Mode coupling III --through Non-linear process--

\[
\rho_0 \frac{\partial^2}{\partial t^2} \nabla \perp^2 U \sim -B_0^2 \cdot \nabla \nabla \left( \nabla \perp^2 U \right) + \nabla \perp^2 U \times \nabla \nabla \left( \nabla \perp^2 U \right)
\]

\[
\rho_0 \frac{\partial^2}{\partial t^2} \sum_{mn} X_{mn} \exp(i(m\theta - n\varphi - i\omega t)) \sim -B_0^2 \cdot \nabla \left( \sum_{mn} \left( \sum_{m'n'} X_{m'n'} \exp(i(m'\theta - n'\varphi - i\omega t)) \right) X_{mn} \exp(i(m\theta - n\varphi - i\omega t)) \right)
\]

Even if \( B_0 \) (equilibrium field) is homogeneous case, mode coupling happens because perturbed terms are multiplied.

\(
\Rightarrow X_{mn} \exp(i(m\theta-n\varphi)-i\omega t)
\)

is not independent each other.
Method of mode structure analyzing I

The 1st order momentum equations are as follows:

\[
\frac{\partial \rho_1}{\partial t} + \mathbf{v}_1 \cdot \nabla \rho_0 = 0,
\]

\[
\frac{\partial p_1}{\partial t} + \mathbf{v}_1 \cdot \nabla p_0 = 0
\]

Here \( \nabla \cdot \mathbf{v}_1 \) is assumed.

\[
\xi \equiv \frac{\partial \mathbf{v}_1}{\partial t}
\]

\[
\rho_1 + \xi_1 \cdot \nabla \rho_0 = 0,
\]

\[
p_1 + \xi_1 \cdot \nabla p_0 = 0
\]

\[
\rho_1 = -\xi_{1r} \frac{d\rho_0}{dr},
\]

\[
p_1 = -\xi_{1r} \frac{dp_0}{dr}
\]

\[
\xi_{1r} = \rho_1 \left( \frac{d\rho_0}{dr} \right),
\]

\[
\xi_{1r} = p_1 \left( \frac{dp_0}{dr} \right)
\]
Example of observation of mode structure

Evolution of the ECE perturbation
Toroidal Alfven freq. \( \sim 5.3 \times 10^6 \text{Hz} @ 2.75 \text{T} \), \( n_e = 10^{19} \text{m}^{-3} \), \( R_0 = 3.6 \text{m} \)
According to theoretical prediction, the growth rate is around \( \sim 1.6 \times 10^4 \text{Hz} \) (63\( \mu \text{s} \)).

There is discrepancy between the prediction and observation in the growth rate.

The prediction of the ideal interchange mode is quite consistent with the observation on the mode structure.

Method of mode structure analyzing II --tearing case--

\[ p_0 + p_1 \]

\[ p_1 = \hat{p}_1 \cos(m\pi - n\phi) \]

\[ p_1 = \hat{p}_1 \cos(m\pi - n\phi + \pi) \]

Around O point

Around X point

\[ \pi \]
Characteristics MHD equil. related to stability in LHD

Magnetic hill exists in the finite beta gradients region

=>

MHD instabilities (interchange/pressure driven) would appear in high beta regime.

\[ \Lambda_p = 6.2, \ p \sim (1-\rho^2)(1-\rho^8) \]
Characteristics MHD instability in LHD

Fig. 1. Temporal changes of the averaged beta; plasma current normalized by toroidal field; electron density; electron pressure at \( \rho = 0.2, 0.4, 0.6, 0.8, \) and 1.0; and amplitude of \( m/n = 2/1 \) mode.

Fig. 4. Temporal changes of (a) \( \langle \beta_{\text{dia}} \rangle \) and \( n_e \), (b) frequencies of \( m/n = 1/1 \) and 2/3 modes, and amplitudes of (c) \( m/n = 1/1 \) and (d) \( m/n = 2/3 \) modes in typical high-\( \beta \) discharge.
**Quasi-steady high-\( \beta \) discharge**

\(< \beta > = 4.8 \% \)

- No disruptive high beta plasma is maintained during more than 80\( \tau_E \)
- Large shaftanov shift \( \Delta/a_p \sim 40\% \)
- Low-n,m MHD activities
  - No observation of core resonant modes.
  - Only resonating mode with peripheral surf. \((m/n = 2/3 \text{ and } 1/1)\) appear
- Global confinement property is almost same with ISS95 scaling.

\( \tau_E \sim 13.3 \text{ms} \)

No large \( \beta \) flattening enough to affect a global confinement

Small flattening and asymmetric structures are observed
Comparison between peripheral pressure gradient and the prediction of linear MHD stability analysis

δ/aₚ ~ 3% (Ideal)

~ 5% (S=10⁶) consistent with exp.

Calc. by FAR3D

Radial structure of low-n ideal and resistive MHD mode

In <β dia> ~ 4% plasmas, the global MHD mode is predicted unstable, but its radial mode width is narrow (~ 5% of aₚ / growth rate γ/ω_A ~ 10⁻²)

Observed kinetic beta gradients and a contour of growth rate of low-n ideal MHD mode

No strong reduction of gradients

# The gradients are averaged for Δρ=0.1.

even in the mode is expected linearly unstable, when the mode width is narrow, the effect on the confinement is quite small
Example of \( m/n=1/1 \) MHD instability in marginally stable discharge

Inducing impurity gas-puffing => S and grad p changing magnetic config. => \( \partial i/\partial r \) and \( \kappa_n \) around the resonant surface =>
Achievement of marginally unstable discharges with \( \langle \beta \rangle \sim 1\% \) but similar \( b^\% \) level of \( \langle \beta \rangle \sim 5\% \)

\[ \# 92000, \ R_{ax}^V \ 3.75m, \ 0.9T \]

\[ \text{1.87s} \]

After disappearance of Mag. fluc., the beta increases

\( m/n=1/1 \) mag.fluc. is dominant in power spec..
Fluc. level ~0.03%
=> Same order with \( \langle \beta \rangle \sim 5\% \) discharge
Characteristics of m/n=1/1 mode structure

No phase inversion (No mag. island structure?)
=> Similar mode structure to linear theory prediction of res. interchange MHD insta.

Radial profile of SXR fluctuation amplitude
t=1.8s

Line-integrated

Sight lines of SX

Radial profile of SXR fluctuation amplitude

m/n 3/3 2/2 1/1

Coh. in mag. fluc.

Rotation freq. (kHz)

0 2 4 6 8 10

(nanxnan)/20

Sight lines of SX

<β> (%)

0 0.5 1.0 1.5

(kHz)

1.5 2.0 2.5

time (s)

n_e (10^{13} \text{m}^{-3})

NBI

# 92000, \( R_{ax} = 3.75 \text{m}, 0.9 \text{T} \)

\( f \approx 1.8 \text{kHz} \)

Radial profile of SXR fluctuation amplitude

1.8s

Sight lines of SX

Radial profile of SXR fluctuation amplitude

1.8s

Sight lines of SX

Radial profile of SXR fluctuation amplitude

1.8s
Degradation Area due to the m/n=1/1 mode estimated from Te profile

$B_0 = 0.9T$, $b/B_0 \sim 0.03\%$

 Decrease in Te is restricted to peripheral region around resonant rational surface by the m/n=1/1 mode

=> Impact on core region is quite small.
Estimation of m/n=1/1 mode internal structure based on Abel inv. and fitting method

- Observation
- Abel; (line-integrated)
- fitted; (line-integrated)

\[ \Delta_{1/2}/a_p \sim 15\% \]

\[ \Delta/a_p \sim 6\% \]

\[ \xi_r/a_p \equiv \frac{\delta i_{SX}}{d_i_{SX}/d\rho} \]

\[ \Delta/a_p \sim 15\%, \max(\delta i/I_{SX-max}) \sim 3\% \]

in line-averaged fluc.

=>

Radial displacement

\[ \Delta/a_p \sim 6\%, \xi_r/a_p \sim 4\% \]

FWHM in local fluc. amplitude; less than half in line-averaged fluc. amp
Estimation of transport based on modeled perturb.

Effect on confinement? small?? Not clear yet!!

\[ \frac{\xi_r}{a_p} \text{ max} / a_p \sim 6\%, \Delta/a_p \sim 5\% \]

\[ \frac{\xi_r}{a_p} \text{ max} / a_p \sim 10\%, \Delta/a_p \sim 8\% \]
Effect of m/n=1/1 MHD mode on confinement

Confinement performance normalized by an empirical $\tau_E$ scaling (ISS04) in presence and disappearance

After disappearance of Mag. fluc., the beta increases

~10% degradation of the normalized global conf. time by a empirical scaling (ISS04) in the presence of the m/n=1/1 mode
No fatal effect of “global” mode on the helical plasma?

Helical coil of LHD consists of 3 layers. By changing the current ratio in the 3 layers, plasma aspect ratio, mag. shear and mag. hill height are controlled.

High aspect configuration (a special config.) has low magnetic shear and high magnetic hill in LHD

=> Interchange mode is more unstable

Magnetic curvature

Rotaional transform

![Graphs showing magnetic curvature and rotational transform](image)

The magnetic shear of high aspect. conf. is much smaller than that of medium aspect. conf., and $\kappa_n$ in both aspect ratio is almost same at the $m/n=1/1$ rational surface.
\( m/n = 1/1 \) mode in high aspect config. (low shear/high hill)

A collapse occurs in a high aspect plasma
Before the collapse occurs, stability condition of global MHD mode is strongly violated.
$m/n = 1/1$ mode in high aspect config. (low shear/high hill)

A collapse occurs in a high aspect plasma
Before the collapse occurs, stability condition of global MHD mode is strongly violated.
Mode width is much important for the effect on confinement!
## Characteristics of m/n = 1/1 mode in LHD

Several differences of characteristics of the mode in different configurations.

<table>
<thead>
<tr>
<th>Experiments</th>
<th>“non-rotating” mode (High-$A_p$, and/or large $I_p$)</th>
<th>“rotating” mode (high-$\beta$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>radial location</td>
<td>$\rho \sim 0.7$ (currentless)</td>
<td>$\rho \sim 0.9$</td>
</tr>
<tr>
<td>configuration</td>
<td>weak shear, magnetic hill ($D_R&gt;0$)</td>
<td>magnetic hill ($D_R&gt;0$)</td>
</tr>
<tr>
<td>Prediction</td>
<td>Ideal unstable with large mode width</td>
<td>Ideal stable, or unstable with narrow mode width</td>
</tr>
<tr>
<td>frequency</td>
<td>DC ~ several Hz</td>
<td>several kHz</td>
</tr>
<tr>
<td>spatial location</td>
<td>$\phi \sim -120$ deg (near natural error field)</td>
<td>rotating</td>
</tr>
<tr>
<td>S dependence Interaction with static 1/1 island</td>
<td>Low-S =&gt; not appears(?!?)</td>
<td>Low-S =&gt; large signal</td>
</tr>
<tr>
<td></td>
<td>Suppression or growth</td>
<td>Reduction of rotation, suppression</td>
</tr>
</tbody>
</table>

“Ideal” mode

“Resistive” mode